

Problem 2.22

[Difficulty: 3]

2.22 Consider the velocity field $V = ax\hat{i} + by(1 + ct)\hat{j}$, where $a = b = 2 \text{ s}^{-1}$ and $c = 0.4 \text{ s}^{-1}$. Coordinates are measured in meters. For the particle that passes through the point $(x, y) = (1, 1)$ at the instant $t = 0$, plot the pathline during the interval from $t = 0$ to 1.5 s. Compare this pathline with the streamlines plotted through the same point at the instants $t = 0, 1$, and 1.5 s.

Given: Velocity field

Find: Plot of pathline of particle for $t = 0$ to 1.5 s that was at point (1,1) at $t = 0$; compare to streamlines through same point at the instants $t = 0, 1$ and 1.5 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = ax$ $a = 2 \frac{1}{s}$ $v_p = \frac{dy}{dt} = b \cdot y \cdot (1 + c \cdot t)$ $b = 2 \frac{1}{s}$ $c = 0.4 \frac{1}{s}$

So, separating variables $\frac{dx}{x} = a \cdot dt$ $dy = b \cdot y \cdot (1 + c \cdot t) \cdot dt$ $\frac{dy}{y} = b \cdot (1 + c \cdot t) \cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = a \cdot t$ $x_0 = 1 \text{ m}$ $\ln\left(\frac{y}{y_0}\right) = b \cdot \left(t + \frac{1}{2} \cdot c \cdot t^2\right)$ $y_0 = 1 \text{ m}$

Hence $x(t) = x_0 \cdot e^{a \cdot t}$ $y(t) = e^{b \cdot \left(t + \frac{1}{2} \cdot c \cdot t^2\right)}$

Using given data $x(t) = e^{2 \cdot t}$ $y(t) = e^{2 \cdot t + 0.4 \cdot t^2}$

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot y \cdot (1 + c \cdot t)}{a \cdot x}$

So, separating variables $\frac{dy}{y} = \frac{b \cdot (1 + c \cdot t)}{a \cdot x} \cdot dx$ which we can integrate for any given t (t is treated as a constant)

Hence $\ln\left(\frac{y}{y_0}\right) = \frac{b}{a} \cdot (1 + c \cdot t) \cdot \ln\left(\frac{x}{x_0}\right)$

The solution is $y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot (1 + c \cdot t)}$

For $t = 0$ $y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot (1+c \cdot t)} = x$ $t = 1$ $y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot (1+c \cdot t)} = x^{1.4}$ $t = 1.5$ $y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot (1+c \cdot t)} = x^{1.6}$

